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A NUMERICAL SOLUTION OF THE MATRIX RICCATI EQUATIONS

By

Killion Noh

January 20, 1972

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A NUMERICAL SOLUTION OF THE MATRIX RICCATI EQUATIONS

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Submitted in partial fulfillment of the requirements for the degree of Master of Science in Computer Science in the Graduate College of the University of Illinois at Urbana-Champaign and supported in part by the Advanced Research Projects Agency of the Department of Defense and was monitored by the U.S. Army Research Office-Durham under Contract No. DAHCO4 72-C-0001.



ABSTRACT

The eigenvector solution of the time-invariant matrix Riccati equation is discussed. The coefficient matrix of the canonical equation is allowed to have multiple eigenvalues, namely, the matrix could be either derogatory or defective. The solution of matrix Riccati equation is then calculated from a part of similarity transformation which should reduce the coefficient matrix to the Jordan canonical form.



ACKNOWLEDGEMENT

The author would like to express his thanks to his advisor, Professor

Daniel L. Slotnick, and to the ILLIAC IV Project and the Center for Advanced

Computation under whose employ this research was conducted. A note of

appreciation is also extended to the Department of Computer Science for its

cooperation and support during the time of the author's studies at the

University of Illinois.

Deepest gratitude also goes to Professor Ahmed H. Sameh for suggesting this thesis topic and for his guidance throughout its preparation. Without his encouragement and patience, this work would have never been done.



TABLE OF CONTENTS

		Page
1.	INTRODUCTION	1
2.	FORMULATION OF THE PROBLEM	2
3.	SOLUTION OF THE MATRIX RICCATI EQUATION	6
4.	COMPUTATIONAL METHODS AND NUMERICAL RESULTS	13
LIST	of references	18
APPI	ENDIX I. ALGOL PROGRAMS	20
APPI	ENDIX II. ILLIAC IV PROGRAMS	28



1. INTRODUCTION

It is well known [1,2] that the linear regulator problem with quadratic cost functional is reduced to the problem of the matrix Riccati equation.

Although the solution of the matrix Riccati equation exists and is unique [2,3], it is not easy to obtain a numerical solution for the high-order system.

One of the numerical methods is the eigenvector solution which has been studied by several authors [4,5,6]. In their studies, however, the coefficient matrix of the canonical equation is assumed to have distinct eigenvalues, which is a restriction to the method of eigenvector solutions.

Recently, Martensson [7] discussed the case in which the coefficient matrix is nondiagonable. Similarly, in this paper, the assumption of distinct eigenvalues will also be removed so that the coefficient matrix is allowed to have the multiple eigenvalues, namely, the matrix is either derogatory or defective. To do this, we explicitly construct the similarity transformation which should reduce the coefficient matrix to the Jordan canonical form, either diagonal or nondiagonal, and then the solution of the matrix Riccati equation is calculated from a part of the similarity transformation matrix.

Since matrix Riccati equations of fairly large sizes appear in many engineering applications and since the solution algorithm can be easily constructed such that it becomes suitable for a parallel computer [8], we provide in Appendix II the basic codes written in ILLIAC IV language, GLYPNIR.

2. FORMULATION OF THE PROBLEM

In this section, the source of the matrix Riccati equation and its relation to optimal control problems are summarized in short for our further discussions [1,2,3].

Let us consider the linear time-invariant dynamical system

$$x(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

$$x(t_0) = x_0$$
(2.1)

where A, B, and C are n x n, n x r, and m x n constant matrices, respectively, and x(t) is an n-dimensional state vector, u(t) is an r-dimensional control vector, and y(t) is an m-dimensional output vector, respectively. Further, we assume that the system (2.1) is completely observable. The optimal output regulator problem is then to determine the control u(t) which minimizes the quadratic cost functional,

$$V_1 = \frac{1}{2} y^T (t_f) Py(t_f) + \frac{1}{2} \int_{t_0}^{t} (y^T(t) Qy(t) + u^T(t)Ru(t))dt$$
(2.2)

where P and Q are m x m symmetric positive semidefinite matrices and R is an r x r symmetric positive definite matrix. Substituting y(t) = Cx(t) into (2.2), V_{q} is rewritten as

$$V_{1} = \frac{1}{2} x^{T}(t_{f}) C^{T}PCx(t_{f}) + \frac{1}{2} \int_{t_{0}}^{t_{f}} (x^{T}(t)C^{T}QC x(t) + u^{T}(t)Ru(t))dt$$
(2.3)

Since the system (2.1) is completely observable, $C^{T}PC$ and $C^{T}QC$ are symmetric and positive semidefinite [1].

The optimal feedback control, therefore, is given by

$$u^*(t) = -R^{-1}B^TK(t)x(t)$$
 (2.4)

where K(t) is an n x n symmetric positive definite matrix which is the

solution of the matrix Riccati equation

$$K(t) + K(t) A + A^{T}K(t) - K(t)BR^{-1}B^{T}K(t) + C^{T}QC = 0$$
 (2.5)

with the boundary condition

$$K(t_f) = C^{T}PC$$
 (2.6)

Furthermore, the optimal trajectory is the solution of the system of differential equations

$$\dot{x}(t) = (A - BR^{-1}B^{T}K(t)) x(t)$$

$$x(t_{0}) = x_{0}$$
(2.7)

The minimum cost is given by

$$V_1^* = \frac{1}{2} x^T(t) K(t) x(t)$$
 (2.8)

In addition to the assumption of complete observability we further assume that the system is completely controllable. Setting P=0 and $t_f=\infty$, the output regulator problem turns out to be the following:

minimize

$$V = \frac{1}{2} \int_0^\infty (y^T(t)Qy(t) + u^T(t)Ru(t)) dt$$
 (2.9)

subject to

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

$$x(0) = x_0$$
(2.10)

Then the optimal control is given by

$$u(t) = -R^{-1}B^{T}K x(t)$$
 (2.11)

where K is the n x n symmetric positive definite matrix which is the solution of algebraic Riccati equation

$$KA + A^{T}K - KBR^{-1}B^{T}K + C^{T}QC = 0$$
 (2.12)

The optimal trajectory is the solution of the system

$$\dot{x}(t) = Gx(t)$$

$$G = A - BR^{-1}B^{T}K$$

$$x(0) = x_{0}$$
(2.13)

Note that for an optimal control system the matrix G is diagonable and has eigenvalues with negative real parts. Therefore, the symmetric positive definite solution K of (2.12) must be such that G has eigenvalues with negative real parts.

Kalman [2] has shown that if the system (2.1) is completely observable and completely controllable, then

$$\lim_{\substack{t \to \infty \\ f}} K(t_f) = K \tag{2.14}$$

We shall observe this equation in the following.

Let
$$K(t) = Z(t)X^{-1}(t)$$
 (2.15)

where X(t) and Z(t) are n x n matrices and X(t) is non-singular. Substituting

$$\dot{K}(t) = (\dot{Z}(t) - K(t) \dot{X}(t)) X^{-1}(t)$$
(2.16)

into (2.5) and setting

$$\dot{X}(t) = AX(t) - BR^{-1}B^{T}Z(t)$$
 (2.17)

we obtain

$$\dot{Z}(t) = -C^{T}QCX(t) - A^{T}Z(t)$$
 (2.18)

Therefore (2.5) becomes a system of 2n-dimensional linear homogeneous differential equations

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}} \\ -\mathbf{c}^{\mathrm{T}}\mathbf{Q}\mathbf{C} & -\mathbf{A}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{z}(t) \end{bmatrix}$$
(2.19)

Since we are interested in increasing the time t, let

$$\tau = t_{f} - t \tag{2.20}$$

Then

where

$$W = \begin{bmatrix} -A & BR^{-1}B^{T} \\ C^{T}QC & A^{T} \end{bmatrix}$$

with

$$X(0) = I$$

$$Z(0) = CTQC$$
(2.22)

Therefore, the solution will be given by the following matrix exponential:

$$\begin{bmatrix} X(\tau) \\ Z(\tau) \end{bmatrix} = e^{W\tau} \begin{bmatrix} I \\ C^{T}QC \end{bmatrix}$$
 (2.23)

Consequently, the solution of the matrix Riccati equation will be obtained by (2.15) as $\tau \to \infty$. In the next section, we shall show that two matrices $X(\tau)$ and $Z(\tau)$ are obtained from the similarity transformation which reduces W into the Jordan canonical form. Throughout the remaining discussions, the increasing time variable t will be used instead of τ .

3. SOLUTION OF THE MATRIX RICCATI EQUATION

We begin with two lemmas for our main result.

Lemma 1. The eigenvalues of the matrix W of (2.21) are symmetric with respect to the imaginary axis.

Proof. Let I_1 be a 2n x 2n matrix given by

$$I_{1} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$
 (3.1)

where I is an n x n identity matrix. Then it is obvious that

$$I_{\gamma}^{-1} = I_{\gamma}^{T} \tag{3.2}$$

$$I_{\gamma}WI_{\gamma}^{T} = -W^{T}$$
 (3.3)

Since the eigenvalues of a matrix are unchanged by either similarity transformations or transposing, the eigenvalues of W occur in pairs with opposite signs.

Lemma 2. Let

$$T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$
 (3.4)

where A_{11} , A_{22} , A_{12} , and A_{21} are n x n, m x m, n x m, and m x n matrices, respectively. Assume that A_{11} is non-singular. Then

$$T^{-1} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$
 (3.5)

exists iff $A_{22} - A_{21} A_{11} A_{12}$ is non-singular, and is given by

$$T_{11} = A_{11}^{-1} + A_{11}^{-1} A_{12} T_{22} A_{21} A_{11}^{-1}$$

$$T_{12} = -A_{11}^{-1} A_{12} T_{22}$$

$$T_{21} = -T_{22} A_{21} A_{11}^{-1}$$

$$T_{22} = (A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1}$$

$$(3.6)$$

where submatrices T_{11} , T_{22} , T_{12} , and T_{21} are the same sizes as A_{11} , A_{22} , A_{12} , and A_{21} , respectively.

Proof. Suppose $A_{22} - A_{21} A_{11}^{-1} A_{12}$ is non-singular. From $TT^{-1} = I$, $A_{11} T_{11} + A_{12} T_{21} = I_n$ $A_{11} T_{12} + A_{12} T_{22} = 0$ $A_{21} T_{11} + A_{22} T_{21} = 0$ (3.7)

$$A_{21} T_{12} + A_{22} T_{22} = Im$$

Premultiplying the second equation by A_{21} A_{11} and subtracting from the fourth,

$$T_{22} = (A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1}$$

And hence

$$T_{12} = -A_{11}^{-1} A_{12} T_{22}$$

Similarly, from the first and third,

$$T_{21} = -T_{22} A_{21} A_{11}^{-1}$$
 $T_{11} = A_{11}^{-1} + A_{11}^{-1} A_{12} T_{22} A_{21} A_{11}^{-1}$

Hence T^{-1} is a right inverse of T. In a similar way we can show that T^{-1} is also a left inverse of T. Conversely, suppose T is non-singular. Then

$$0 \neq \det \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \det \begin{bmatrix} A_{11} & 0 \\ A_{21} & I_{m} \end{bmatrix} \begin{bmatrix} I_{n} & A_{11} & A_{12} \\ 0 & A_{22} - A_{21} & A_{11} & A_{12} \end{bmatrix}$$

$$= \det (A_{11}) \det (A_{22} - A_{21} A_{11}^{-1} A_{12})$$
 (3.8)

Therefore, $A_{22} - A_{21} A_{11} A_{12}$ is nonsingular.

Assuming that W has distinct eigenvalues, we now turn to constructing the similarity transformation with which W of (2.21) is reduced to the

diagonal form. By lemma 1, W has n eigenvalues with positive real parts and n eigenvalues with negative real parts. If we construct the similarity transformation S, whose columns are the right eigenvectors of W, such that the first n columns are the right eigenvectors of W corresponding to the n eigenvalues of W with positive real parts, then

$$S^{-1}WS = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{2n} \end{bmatrix}$$
 (3.9)

where, of course, the rows of S⁻¹ are the left eigenvectors of W.

O'donnell [5] and Potter [6] have shown that if we partition S into four n x n submatrices,

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$
 (3.10)

then the solution of the matrix Riccati equation is given by

$$K = S_{21} S_{11}^{-1} (3.11)$$

provided that S₁₁ is non-singular.

We shall now remove the restriction that W has distinct eigenvalues and present an alternative proof to that of Martensson [7] to show that (3.11) holds for the case in which W has nondiagonal Jordan form.

Let us consider the general Jordan form J with the similarity transformation S, which will be constructed later,

$$s^{-1}Ws = J$$

where,
$$J = \begin{bmatrix} J_{m_1} \\ \ddots \\ J_{m_k} \end{bmatrix}$$
 (3.12)

and in which

If we denote S by its column vectors s_1, \ldots, s_{2n} , then

$$W[s_1, s_2, ..., s_{2n}] = [s_1, s_2, ..., s_{2n}] J$$
 (3.14)

For any Jordan block J_{m_i} , if $m_i > 1$ then, dropping subscripts of s for convenience,

$$Ws^{(1)} = s^{(1)}\lambda_{i}$$

$$Ws^{(2)} = s^{(1)} + \lambda_{i} s^{(2)}$$

$$...$$

$$Ws^{(m_{i})} = s^{(m_{i}-1)} + \lambda_{i} s^{(m_{i})}$$
(3.15)

or

$$(W - \lambda_{i}I) s^{(1)} = 0$$

$$(W - \lambda_{i}I)^{2} s^{(2)} = 0$$

$$(W - \lambda_{i}I)^{m} i s^{(m}i) = 0$$

$$(3.16)$$

Thus, $s^{(j)}$, $1 \le j \le m_i$, is a principal vector of degree j associated with λ_i . Since S is nonsingular, the m_i principal vectors $s^{(1)}$, ..., $s^{(m_i)}$ are independent. If we consider all the Jordan blocks J_{m_i} , $1 \le i \le \ell$, we have the 2n principal vectors s_1 , s_2 , ..., s_{2n} which constitute the similarity transformation S in (3.12). Further, we rearrange the columns of S such that the first n vectors are the principal vectors corresponding to the n eigenvalues with positive real parts of W.

Let us now split J of (3.12) into

$$J = \begin{bmatrix} J_1 & \vdots & \vdots & \vdots & \vdots \\ \vdots & J_2 \end{bmatrix} + \begin{bmatrix} E_1 & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & E_2 \end{bmatrix}$$
 (3.17)

where J_1 and J_2 are diagonal with the eigenvalues of W with positive and negative real parts respectively, and E_1 and E_2 are matrices with one's on the super-diagonals and zeros elsewhere. To associate with (3.6), let us partition S and S⁻¹ as

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$
(3.18)

where all submatrices are n x n. Note that, by lemma 2, it is necessary to assume that S_{11} and S_{22} - S_{21} S_{12} are nonsingular.

Then

$$e^{Wt} = e^{SJS^{-1}t}$$

$$= Se^{Jt}S^{-1}$$

$$= \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} e^{J_1t} & 0 \\ 0 & e^{J_2t} \end{bmatrix} \begin{bmatrix} e^{E_1t} & 0 \\ 0 & e^{E_2t} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

$$= \begin{bmatrix} S_{11}e^{J_1t}e^{E_1t}T_{11} + S_{12}e^{J_2t}e^{E_2t}T_{21} & S_{11}e^{J_1t}e^{E_1t}T_{12} + S_{12}e^{J_2t}e^{E_2t}T_{22} \\ S_{21}e^{J_1t}e^{E_1t}T_{11} + S_{22}e^{J_2t}e^{E_2t}T_{21} & S_{21}e^{J_1t}e^{E_1t}T_{12} + S_{22}e^{J_2t}e^{E_2t}T_{22} \end{bmatrix}$$

$$(3.19)$$

Since the elements of J_2 have negative real parts, $e^{J_2 t} \rightarrow 0$ as $t \rightarrow \infty$ and thus the terms having $e^{J_2 t}$ in (3.19) vanish as $t \rightarrow \infty$. Therefore,

$$\begin{bmatrix}
X(t) \\
Z(t)
\end{bmatrix} = e^{Wt} \begin{bmatrix} I \\
C^{T}QC \end{bmatrix}$$

$$= \begin{bmatrix}
S_{11}e^{J_{1}t}e^{E_{1}t}T_{11} & S_{11}e^{J_{1}t}e^{E_{1}t}T_{12} \\
S_{21}e^{J_{1}t}e^{E_{1}t}T_{11} & S_{21}e^{J_{1}t}e^{E_{1}t}T_{12}
\end{bmatrix} \begin{bmatrix} I \\
C^{T}QC \end{bmatrix}$$

$$= \begin{bmatrix}
S_{11}e^{J_{1}t}e^{E_{1}t}(T_{11} + T_{12}C^{T}QC) \\
S_{21}e^{J_{1}t}e^{E_{1}t}(T_{11} + T_{12}C^{T}QC)
\end{bmatrix} (3.20)$$

Therefore the solution of the matrix Riccati equation is given by

$$K = \lim_{t \to \infty} Z(t) X^{-1}(t)$$

$$= \lim_{t \to \infty} [S_{21} e^{J_1 t} e^{E_1 t} (T_{11} + T_{12} c^{T_{QC}})][S_{11} e^{J_1 t} e^{E_1 t} (T_{11} + T_{12} c^{T_{QC}})]^{-1}$$

$$= S_{21} S_{11}^{-1}$$
(3.21)

provided that the indicated inverse exists. However, we shall show that only the nonsingularity of S_{11} and S_{22} - S_{21} S_{11} is needed for the stationary solution of the matrix Riccati equation. From (3.12),

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} & \begin{bmatrix} -A & BR^{-1}B^{T} \\ C^{T}QC & A^{T} \end{bmatrix} & \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = J$$
 (3.22)

Equating the (2,1) elements on each side of (3.22),

$$(-T_{21} A + T_{22} C^{T}QC) S_{11} + (T_{21} BR^{-1}B^{T} + T_{22} A^{T}) S_{21} = 0$$

or

$$-T_{21} AS_{11} + T_{22} A^{T} S_{21} + T_{21} BR^{-1}B^{T} S_{21} + T_{22} C^{T}QC S_{11} = 0$$
 (3.23)

Substituting $T_{21} = -T_{22} S_{21} S_{11}^{-1}$ into (3.23),

$$T_{22}S_{21}S_{11}^{-1} + T_{22}A^{T}S_{21} - T_{22}S_{21}S_{11}^{-1}BR^{-1}B^{T}S_{21} + T_{22}C^{T}QCS_{11}^{-1} = 0$$
(3.24)

Premultiplying by T_{22}^{-1} and postmultiplying by S_{11}^{-1} ,

$$S_{21}S_{11}^{-1}A + A^{T}S_{21}S_{11}^{-1} - S_{21}S_{11}^{-1}BR^{-1}B^{T}S_{21}S_{11}^{-1} + C^{T}QC = 0$$
 (3.25)

which is to be proved.

4. COMPUTATIONAL METHODS AND NUMERICAL RESULTS

The major three steps for computing the solution of the matrix Riccati equation are as follows:

- 1) Compute eigenvalues and eigenvectors of W,
- 2) Compute principal vectors of W if W is defective,
- 3) Compute S₂₁ S₁₁ from S.

For computing the eigensystem, we used two methods: (i) the QR algorithm [9] with an inverse iteration routine [10], and (ii) Jacobi-like method [11] for finding the eigenvalues and eigenvectors. Their performances are satisfactory and two numerical test examples are given in Test Result 1 and Test Result 3. When the matrix W is nondiagonable, Test Result 2, the first method requires the evaluation of the principal vectors as well [12]. This poses some numerical difficulties as we are essentially trying to solve the ill-conditioned system of equation

$$(W - \tilde{\lambda}I)^k x_{i+1} = x_i, k > 1$$

where $\tilde{\chi}$ is an approximation of true eigenvalues λ . Thus, the Jacobi-like method is more attractive in this case for providing an approximate eigensystem that is suitable for engineering purposes.

Appendix I contains the listings of the ALGOL programs used on the B-6500. Appendix II contains the listings of two ILLIAC IV programs, namely, the inverse iteration routine and a routine for solving a system of complex linear equations, both written in GLYPNIR. ILLIAC IV programs for the QR algorithm [13, 14] and the Jacobi-like algorithms [15, 16] are already available.

Test Result 1.

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The numerical example was tested by the QR algorithm and the inverse iteration method. The ten distinct eigenvalues of W are

± 1,

± 1.728760477185 ± 1.577533767573i,

± 1.353195784046 ± 1.153749899284i.

We choose the five eigenvectors corresponding to the five eigenvalues with positive real parts and computing $K = S_{21} S_{11}^{-1}$ from them gives the solution K as

1.262782609 2.494009759 -0.819173651 0.668267901 -0.443608958 2.494009759 7.435451164 -1.825741858 1.122910432 -0.668267901 -0.819173651 -1.825741858 1.638347303 1.825741858 -0.819173651 0.668267901 1.122910432 1.825741858 7.435451164 -2.494009759 -0.443608958 -0.668267901 -0.819173651 -2.494009759 1.262782609

All the eigenvalues of $G = A - BR^{-1}B^{T}K$ have negative real parts and hence the above solution K is acceptable.

Test Result 2.

$$A = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P = Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 1 \end{bmatrix}$$

The matrix

$$W = \begin{bmatrix} 3 & -2 & 0 & 0 \\ 2 & -1 & 0 & 1 \\ 0 & 0 & -3 & -2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

is defective, that is, W has four eigenvalues ±1 (double) and each +1 and -1 correspond to a quadratic elementary divisor, respectively. This example was tested both by Jacobi-like method and by a combination of the QR algorithm and the inverse iteration method.

The Jacobi-like method gives K as

-10.0000084 6.0000043

6.0000034 -4.0000027

and the QR and inverse iteration method give K as

-9.9999917 5.9999947

5.9999953 -3.9999983

The exact solution is

$$\begin{bmatrix} -10 & 6 \\ 6 & -4 \end{bmatrix}$$

For computing the principal vector by the QR and inverse iteration, we chose the perturbation for the true eigenvalues as much as 10^{-6} .

Test Result 3.

The exact eigenvalues of W are

Since the matrix W is derogatory, +5 and -5 correspond to a two-dimensional subspace, respectively. The computed eigenvalues by Jacobi-like method are

14.99999999999999999999999999

4.999999999999999999999999999

-4.9999999982 -4.99999999948

The computed solution K is given by

-1.99999999911 1.99999999942 1.99999999993 -2.000000000015

1.99999999884 -2.000000000015 -2.000000000000 2.00000000087

2.00000000276 -2.00000000407 -2.000000000378 2.000000000480

-2.00000000364 2.00000000509 2.000000000480 -2.00000000568

in which the absolute error is less than 10⁻⁹.

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APPENDIX I

ALGOL PROGRAMS

```
* THO ALGOL PROCEDURES, RICCATIL AND RICCATIZ, FOR COMPUTING THE * SOLUTION OF THE MATRIX RICCATI EQUATION ARE LISTED BELOW, FOR
                                                                                    00000100?
                                                                                    00000200?
* "RICCATIL" THE PROCEDURE "EIGEN" (BY EBERLEIN AND BUOTHHOYD) WAS
                                                                                    00000300?
I USED WHILE A COMBINATION OF "HOR" (BY MARTIN, ET AL) AND
                                                                                    00000400?
# HINVITERATION WAS APPLIED TO "RICCATIZ", LINEAR EQUATION SOLVER BY BOWDLER, ET AL, WAS ALSO USED TO PROVIDE THE INVERSE OF A MATRIX
                                                                                    000005007
                                                                                    00000600?
* AND TO CALCULATE THE PRINCIPAL VECTORS FROM ((W=LAMBUA+1)++K)+X=0.
                                                                                    000007007
                                                                                    00000800?
                                                                                    00000900?
PROCEDURE RICCATII(N,L,M,A,B,C,P,Q,R,TMX,W,K);
                                                                                    000010002
VALUE N.L.M.TMXI
                                                                                    00001100?
INTEGER N.L. M. TMX;
                                                                                    000012002
                                                                                    00001300?
ARRAY A.B.C.P.Q.R.W.K[1.1];
COMMENT SOLVES THE MATRIX RICCATI EQUATION
                                                                                    00001400?
           KA+(AT)K=KR(RINV)(BT)K+(CT)QC=0.
                                                                                    00001500?
         A IS N+N+ B IS N+L+ C IS M+N+ P AND Q ARE M+M+ R IS L+L
                                                                                    00001600?
         MATRICES. RESPECTIVELY. FORM 2N+2N MATRIX W AND COMPUTE
                                                                                    000017007
         EIGENVALUES AND EIGENVECTORS OF W. WHERE FOUR NON BLOCKS
                                                                                    000018007
                                                                                    00001900?
         OF W ARE (1.1) = A. (1.2) = B(RINV)(BT). (2.1) = (CI)QC. AND
           .2) = (AT), REARRANGE EIGENVECTOR (PRINCIPAL VECTOR) MATRIX
                                                                                    00002000?
         SO THAT THE FIRST N COLUMNS CORRESPOND TO TO THE ELGENVALUES WITH POSITIVE REAL PARTS. DENOTING THE UPPER HALF UP N COLUMN VECTORS BY SMTX1 AND THE LOWER HALF BY SMTX2. THE SOLUTION
                                                                                    000021002
                                                                                    000022002
                                                                                    000023002
                                                                                    000024007
          IS THEN GIVEN BY K=(SMTx2)(SMTx1INV), THE PRUCEDURE
         "EIGEN" IS USED FOR EIGENPRUBLEM OF WI
                                                                                    000025002
                                                                                    00002600?
                                                                                    000027002
REGIN COMMENT FIRST DECLARE PROCEDURES;
                                                                                    00002800?
PROCEDURE SETUP(NoLomo A & B & C & Q & RINV & W) ;
                                                                                     00002900?
                                                                                    00003000?
VALUE N.L.M. INTEGER N.L.M.
                                                                                    00003100?
REAL ARRAY A.B.C.Q.RINV.N[1.1];
         THE PROCEDURE SETS UP THE 2N+2N MATRIX W. RINY IS INVERSE OF R:00003200?
COMMENT
REGIN
                                                                                     00003300?
  REAL ARRAY RINVBT[1:L. 1:N] . QC[1:M.1:N];
                                                                                    00003400?
                                                                                    00003500?
  REAL SUM;
                                                                                    00003600?
  INTEGER I, J.KI
                                                                                    00003700?
  FOR I:=1 STEP 1 UNTIL N DO
                                                                                    00003800?
  FOR JI=1 STEP 1 UNTIL N DO
                                                                                    00003900?
    BEGIN
                                                                                    00004000?
       *[[.I]A==*[[.I]W
                                                                                    000041002
       III,LJA=1[L+N,I+N]W
                                                                                    00004200?
    END:
                                                                                    00004300?
  FOR I = 1 STEP 1 UNTIL L DO
                                                                                    00004400?
  FOR J:=1 STEP 1 UNTIL N DO
                                                                                    00004500?
    BEGIN SUMIRO. 01
                                                                                    00004600?
       FOR K = 1 STEP 1 UNTIL L DO
                                                                                    00004700?
         SUM: =SUM+RINV(I,K)+B(J,K);
                                                                                    00004800?
       RINVBT[I,J]:=SUM;
                                                                                    00004900?
    END:
                                                                                    000050007
  FOR I = 1 STEP 1 UNTIL N DO
                                                                                    00005100?
  FOR JIMI STEP 1 UNTIL N DO
                                                                                    00005200?
    BEGIN SUMI = 0.01
                                                                                    00005300?
       FOR KIN1 STEP 1 UNTIL L DO
                                                                                    00005400?
         SUM:=SUM+B[I,K]+RINVBT[K,J];
                                                                                     00005500?
       INUZ : [[+N.]]W
                                                                                     00005600?
    END:
                                                                                     00005700?
  FOR Is=1 STEP 1 UNTIL M DO
```

000058007

000059001

000060001

000061001

000062001 000063001

000064001

000065001

000066001

000067001 000068001

000069001

000070001

000071007

000072007

000074007

000076007

000077001

000079007

000081001

000082001

000083007

000084001

000085001

000086001

000087001

000088007

000089007

000090007

000091007

000092007

000093001

000095001

000097007

000098007

000099007

000100007

000101007

000102007

000103007

000104007

000105007

000106007

000107007

000108007

000109007

000110007

000112007

000113007

000114007

```
FOR JEST STEP 1 UNTIL N DO
    BEGIN SUMIRO. 01
      FOR K = 1 STEP 1 UNTIL H DO
         SUMI = SUM+Q[I,K]+C[K,J];
      QC[[.J]; =SUM;
    END:
  FOR I := 1 STEP 1 UNTIL N DO
  FOR JIH1 STEP 1 UNTIL N DO
    BEGIN SUMINO. 01
      FOR K1=1 STEP 1 UNTIL M DO
        SUM : #SUM+C[K,I]+QC[K,J];
      IMUZ=:[[, I+N]W
    END:
END OF SETUPE
COMMENT INSERT THE PROCEDURES INNPROD. DECOMPOSE. AND ACCSOLVE;
PROCEDURE INVERSE(N.A.X);
VALUE N; INTEGER N;
DOUBLE ARRAY A, X(1,1);
COMMENT THE BROCES
        THE PROCEDURE COMPUTES THE INVERSE OF AN NON REAL
        UNSYMMETRIC MATRIX A. USING THE PROCEDURES "DECOMPUSE" AND
        "ACCSOLVE", WRITING ON XI
REGIN
  INTEGER I.J. IT. D2;
  REAL D1:
  ARRAY AA.BB.B[1:N.1:N].P[1:N];
    FUR I . = 1 STEP 1 UNTIL N DO
    FUR JIE1 STEP 1 UNTIL N DO
      IF J=I THEN B[[,J]; =1.0 ELSE B[[,J]; =0.0;
    FUR I =1 STEP 1 UNTIL N DO
    FUR JIE1 STEP 1 UNTIL N DO
      ITA=:[L,I]AA
    DECOMPOSE (N. AA. D1. D2.P);
    ACCSULVE(N.N.A.AA.P.B.X.BB.IT):
END INVERSE
COMMENT INSERT THE PROCEDURE EIGEN:
PROCEDURE MULT(N1.N2.N3.A.B.C);
VALUE NIONZONS: INTEGER NIONZONS:
DOUBLE ARRAY A.B.C[1,1];
REGIN
  INTEGER I.J.KI
  REAL SUM!
  FOR I:=1 STEP 1 UNTIL N1 DO
  FOR J:=1 STEP 1 UNTIL N3 DO
  BEGIN SUM 1=0.01
    FUR K = 1 STEP 1 UNTIL N2 DO
      SUM: SUM + A[I,K] + B[K,J];
    CLI, JI = SUM ;
  ENDI
END MULTI
  ARRAY RINV, SMTX1, SMTX2[1:N, 1:N], WW, TEMP, VECT[1:N+N, 1:N+N]:
  INTEGER ARRAY SEARCH[1:N];
  INTEGER N2. I. J. INDEXI
```

```
N2 1 = N+ N1
                                                                              00011500?
  INVERSE(L.R.RINV);
  SETUP(N, L, M, A, B, C, Q, RINV, W);
                                                                              00011700?
  FOR I; #1 STEP 1 UNTIL N2 DO
                                                                              000118002
  FOR JITI STEP 1 UNTIL NO DO
                                                                              000119002
    TEMP[[,J]:=W[],J];
                                                                              000120002
  EIGEN(N2.THX.TEMP.VECT);
                                                                              00012100?
  COMMENT SEARCH N COLUMNS OF VECT CORRESPONDING TO N LIGENVALUES
                                                                              000122002
          WITH POSITIVE REAL PARTS, SMTX1 AND SMTX2 ARE ITS UPPER
                                                                              00012300?
          AND LOWER HALVES, RESPECTIVELY.
                                                                              000124002
  INDEX:=01
                                                                              00012500?
  FOR I = 1 STEP 1 UNTIL N2 DO
                                                                              000126002
    IF TEMP[I.I] GTR O THEN
                                                                              000127002
    BEGIN
                                                                              00012800?
      INDEX:=INDEX+1:
                                                                              00012900?
      SEARCH[INDEX] != I !
                                                                              000130002
    END:
                                                                              00013100?
  FOR JIM1 STEP 1 UNTIL N DO
                                                                              000132002
  FOR I = 1 STEP 1 UNTIL N DO
                                                                              000133002
  BEGIN
                                                                              000134002
    SMTx1[I,J]; = VECT[I + SEARCH[J]];
                                                                              000135002
    SMTX2[I,J]:=VECT[N+I.SEARCH[J]];
                                                                              00013600?
  FN01
                                                                              00013700?
  INVERSE(N, SMTX1, TEMP);
                                                                              00013800?
  MULI(NONONOSHTX20TEMPOK);
                                                                              000139002
                                                                              00014000?
END RICCATILL
                                                                              00014100?
                                                                              00014200?
PROCEDURE RICCATIZ(NoLOMOAOBOCOPOQOROMOKODEFECTODEROGALE);
                                                                              00014300?
VALUE N.L.M;
                                                                              00014400?
INTEGER N.L.M;
                                                                              00014500?
ARRAY A. 3. C. P.Q. R. W. K[1.1];
                                                                              00014600?
                                                                              00014700?
INTEGER ARRAY DEFECT, DERUGATE[1];
COMMENT SOLVES THE MATRIX RICCATI EQUATION
                                                                              00014800?
          KA+(AT)K-KB(KINV)(BT)K+(CT)QC=0.
                                                                              00014900?
        A IS NOND B IS NOLD C IS MOND P AND Q ARE MOM, R IS LOL
                                                                              00015000?
        MATRICES. RESPECTIVELY. FORM 2N+2N MATRIX W AND COMPUTE
                                                                              00015100?
        EIGENVALUES AND EIGENVECTORS OF W. WHERE FOUR NON BLOCKS
                                                                              00015200?
        OF W ARE (1:1)==A, (1:2)=B(RINV)(BT), (2:1)=(Cf)gC, AND
                                                                              00015300?
                                                                              00015400?
        (2.2)=(AT), REARRANGE EIGENVECTOR(PRINCIPAL VECTOR) MATRIX
        SO THAT THE FIRST N COLUMNS CORRESPOND TO TO THE ELGENVALUES WITH POSITIVE REAL PARTS. DENOTING THE UPPER HALF UP N COLUMN
                                                                              00015500?
                                                                              00015600?
        VECTORS BY SMTX1 AND THE LOWER HALF BY SMTX2. THE SOLUTION
                                                                              000157002
                                                                              00015800?
        K IS THEN GIVEN BY K=(SMTX2)(SMTX1INV), THE PRUCEDURES
                                                                              00015900?
        "HSHLD" "HOR" AND "INVITERATION" ARE USED FOR EIGENPROBLEM
                                                                              00016000?
        OF WI
                                                                              00016100?
                                                                              00016200?
REGIN COMMENT FIRST DECLARE PROCEDURES;
                                                                              00016300?
                                                                              00016400?
PROCEDURE SETUP(Nalama AaBa CagaRINV, W);
                                                                              00016500?
VALUE N.L.M; INTEGER N.L.M.
REAL ARRAY A.B.C.Q.RINV.W[1:1];
                                                                              00016600?
COMMENT THE PROCEDURE SETS UP THE 2N+2N MATRIX W. RINV IS INVERSE OF R; 00016700?
                                                                              00016800?
PFGIN
                                                                              00016900?
  REAL ARRAY RINVBT[1:L.1:N].QC[1:M.1:N];
                                                                              00017000?
  REAL SUMI
                                                                              00017100?
  INTEGER I.J.KI
```

```
000172002
  FOR I:=1 STEP 1 UNTIL N DO
                                                                               000173007
  FOR J:=1 STEP 1 UNTIL N DO
                                                                               000174002
    BEGIN
      #[[,I]A==:[L,I]W
                                                                               000175002
                                                                               000176002
      ILI . LJA = I [L+N , I+N] W
                                                                               00017700?
    END:
                                                                               000178002
  FOR I = 1 STEP 1 UNTIL L DO
  FOR J:=1 STEP 1 UNTIL N DO
                                                                               000179002
                                                                               000180007
    BEGIN SUM:=0.0:
      FOR Kimi STEP 1 UNTIL L DO
                                                                               00018100?
        SUM:=SUM+RINV[I,K]+B[J,K];
                                                                               00018200?
                                                                               000183007
      RINVBT[I,J]:=SUM;
                                                                               000184007
    END:
                                                                               000185007
  FOR I:=1 STEP 1 UNTIL N DO
  FOR J:=1 STEP 1 UNTIL N DO
                                                                               000186007
                                                                               000187007
    BLGIN SUM:=0.0;
      FOR K = 1 STEP 1 UNTIL L DO
                                                                               000188002
        SUM:=SUM+B[I,K]+RINVBT[K,J];
                                                                               000189007
                                                                               00019000?
      I N+J] 1 = SUM 1
    END:
                                                                               00019100?
  FOR I:=1 STEP 1 UNTIL M DO FOR J:=1 STEP 1 UNTIL N DO
                                                                               000192007
                                                                               00019300?
    BEGIN SUM := 0.0;
                                                                               000194002
                                                                               00019500?
      FOR K:=1 STEP 1 UNTIL M DO
        SUM:=SUM+Q[I,K]+C[K,J];
                                                                               000196007
      QC[I+]] := SUM;
                                                                               00019700?
    ENDI
                                                                               00019800?
  FOR I:=1 STEP 1 UNTIL N DO
                                                                               000199002
  FOR J:=1 STEP 1 UNTIL N DO
                                                                               00020000?
    BEGIN SUM = 0.01
                                                                               00020100?
      FOR K:=1 STEP 1 UNTIL M DO
                                                                               00020200?
                                                                               00020300?
        SUM:=SUM+C[K,I]+QC[K,J];
      IMUZ=:[L,I+N]W
                                                                               00020400?
    END;
                                                                               00020500?
END OF SETUPE
                                                                               00020600?
                                                                               00020700?
COMMENT INSERT THE PROCEDURES INNPROD, DECOMPOSE, AND ACCSOLVE;
                                                                               000208002
                                                                               00020900?
PROCEDURE INVERSE(N.A.X);
                                                                               00021000?
VALUE NI INTEGER NI
                                                                               00021100?
DOUBLE ARRAY A.X[1,1]:
                                                                               00021200?
COMMENT THE PROCEDURE COMPUTES THE INVERSE OF AN NON REAL
                                                                               00021300?
        UNSYMMETRIC MATRIX A. USING THE PROCEDURES "DECOMPUSE" AND
                                                                               000214007
        "ACCSOLVE", WRITING ON X;
                                                                               00021500?
REGIN
                                                                               00021600?
  INTEGER I.J. IT. D2;
                                                                               00021700?
  REAL DI:
                                                                               00021800?
  ARRAY AA+BB+B[1:N+1:N]+P[1:N];
                                                                               000219007
    FUR I = 1 STEP 1 UNTIL N DO
                                                                               00022000?
    FUR JIE1 STEP 1 UNTIL N DO
                                                                               00022100?
      IF J=I THEN B[I,J] := 1.0 ELSE B[I,J] := 0.0;
                                                                               00022200?
    FUR I = 1 STEP 1 UNTIL N DO
                                                                               000223007
    FUR JET STEP 1 UNTIL N DO
                                                                               00022400?
      AALI.JJ:=A[I.J];
                                                                               00022500?
    DECOMPOSE (N. AA. D1. D2.P);
                                                                               00022600?
    ACCSULVE(N, N, A, AA, P, B, X, BB, IT);
                                                                               000227007
END INVERSE!
                                                                               00022800?
```

```
00022900?
COMMENT INSERT THE PROCEDURES HIGHLD AND HORS
                                                                                00023000?
                                                                                00023100?
PROCEDURE MULT(N1.N2.N3.A.B.C);
                                                                                00023200?
VALUE NI.NZ.N3; INTEGER NI.NZ.N3;
                                                                                00023300?
DOUBLE ARRAY A.B.C[1.1];
                                                                                000234002
REGIN
                                                                                00023500?
  INTEGER I. J.KI
                                                                                00023600?
  REAL SUM!
                                                                                00023700?
  FOR I: #1 STEP 1 UNTIL N1 DO
                                                                                00023800?
  FOR J: 1 STEP 1 UNTIL N3 DO
                                                                                00023900?
  BEGIN SUM: =0.01
                                                                                00024000?
    FUR KIN1 STEP 1 UNTIL N2 DO
                                                                                000241002
      SUM: #SUM+A[I,K]+B[K,J];
                                                                                000242002
    C[I,J] := SUM;
                                                                                00024300?
  END:
                                                                                00024400?
END MULTI
                                                                                00024500?
                                                                                00024600?
PROCEDURE INVITERATION (Now X x KK + P) :
                                                                                000247002
VALUE N. INTEGER N.KKI
                                                                                000248002
DOUBLE ARRAY W.X[1.1]
                                                                                 000249002
ARRAY PETTI
                                                                                00025000?
COMMENT SOLVES WX=0 BY INVERSE ITERATION, WHERE W IS AN N*N REAL MATRIX, WRITING SOLUTION ON X. THUS, THE PROCEDURE COMPUTES
                                                                                00025100?
                                                                                00025200?
                                                                                00025300?
         THE EIGENVECTORS CORRESPONDING TO THE GIVEN EIGENVALUES.
         N IS THE ORDER OF W AND KK IS THE NUMBER OF ITERATIONS
                                                                                00025400?
                                                                                00025500?
         NEEDED FOR ITERATING EACH EIGENVECTORS
                                                                                 00025600?
REGIN
                                                                                000257002
  INTEGER I, J. DZ. R. L. MAXINDEX;
  DOUBLE DI , MAXVAL ;
                                                                                000258002
                                                                                000259002
  DOUBLE ARRAY WW(1:N,1:N).B.BB(1:N,1:1):
  LABEL INVITADUTE
                                                                                000260007
                                                                                00026100?
  FOR I = 1 STEP 1 UNTIL N DO
                                                                                00026200?
  FOR J:=1 STEP 1 UNTIL N DO
                                                                                00026300?
    :[[.]]W=:[[.]]WW
                                                                                00026400?
  DECUMPOSE (N. WW. D1. D2.P);
                                                                                00026500?
  FOR I = 1 STEP 1 UNTIL N DO
                                                                                00026600?
    Bil, 1 ] != 1.0!
                                                                                000267002
  KK:=1;
         R:=1:
                                                                                00026800?
INVITE
  ACCSOLVE(N.R. W. WW. P. B. X. BB. L) &
                                                                                00026900?
                                                                                00027000?
  MAXVAL 8 = 0 :
                                                                                000271002
  FOR I:=1 STEP 1 UNTIL N DO
                                                                                00027200?
    IF ABS(X[I.1]) GTR MAXVAL THEN
                                                                                00027300?
      BEGIN
                                                                                00027400?
        MAXVAL = ABS(X[I=1]);
                                                                                00027500?
        MAXINDEXIETS
                                                                                00027600?
      END:
                                                                                00027700?
    MAXVAL :=X[MAXINDFX:1];
                                                                                000278002
  FOR I = 1 STEP 1 UNTIL N DO
                                                                                00027900?
    XII,1] = X[I,1]/MAXVAL;
                                                                                00028000?
  FOR I = 1 STEP 1 UNTIL N DO
                                                                                00028100?
    IF ABS(B[I,1]=x[I,1]) GTR #=12 THEN
                                                                                00028200?
    BEGIN
                                                                                00028300?
      IF KK GTR 10 THEN GO TO OUT !
                                                                                00028400?
      FOR J = 1 STEP 1 UNTIL N DU
                                                                                00028500?
         B[J,1]:=X[J,1];
```

```
00028600?
      KK#mKK+13
                                                                                000287007
      Go to INVITA
                                                                                00028800?
    END:
                                                                                000289002
OUT
                                                                                000290002
END INVRESETTA
                                                                                000291002
  INTEGER I, J, JJ, CNT, N2, INDEX, ITCOUNT;
                                                                                000292002
  ARRAY TEMPOORGWOSMTXOVECTOR[1:N+No1:N+N]ORINVOSMTX1,SMTX2[1:No1:N]o
                                                                                000293002
         EIGR · EIGI · INTCH[1:N+N];
                                                                                00029400?
  REAL PERTURB . MPLIER ;
                                                                                000295002
  INTEGER ARRAY SEARCH[1:N];
                                                                                000296002
  N21 = N+N1
                                                                                000297002
  INVERSE(L, R, RINV);
                                                                                000298002
  SFTUP(N,L,M,A,B,C,Q,RINV,W);
                                                                                00029900?
  FOR I := 1 STEP 1 UNTIL N2 DO
                                                                                000300002
  FOR J: #1 STEP 1 UNTIL N2 DO
                                                                                000301002
    ILMP[[,J]I=W[I,J];
                                                                                00030200?
  HSHLD(1, N2, TEMP);
HQR(N2, TEMP, EIGR, EIGI, CNT);
                                                                                000303007
                                                                                000304002
                                                                                00030500?
  FOR I:=1 STEP 1 UNTIL N2 DO
  FOR J:=1 STEP 1 UNTIL N2 00
                                                                                00030600?
    DKGW[[,J]:=W[I,J];
                                                                                00030700?
  COMMENT COMPUTE EIGENVECTORS OR PRINCIPAL VECTORS;
                                                                                00030800?
  FOR JJI=1 STEP 1 UNTIL N2 DO
                                                                                000309002
  REGIN
                                                                                00031000?
    FUR I:=: STEP 1 UNTIL N2 DO
                                                                                000311007
    FUR J:=1 STEP 1 UNTIL N2 DO
                                                                                00031200?
      W[I,J]:=DRGW[I,J];
                                                                                00031300?
    FUR I:=1 STEP 1 UNTIL N2 DO
                                                                                00031400?
      W[I,I] = W[I,T] = EIGR[JJ] + PERTURB;
                                                                                00031500?
    IF JJ=DEFECT[JJ] THEN
                                                                                00031600?
      BEGIN
                                                                                000317007
        FOR I := 1 STEP 1 UNTIL N2 DO
                                                                                00031800?
           W[I,I] # #W[I,I] + PERTURB + MPLIER;
                                                                                00031900?
        MULT(N2, N2, N2, W, W, TEMP);
                                                                                00032000?
        FOR Ism1 STEP 1 UNTIL N2 DO
                                                                                000321002
        FOR JIE1 STEP 1 UNTIL N2 DO
                                                                                00032200?
           W[I.J] = TEMP[I.J] W
                                                                                000323002
      END:
                                                                                00032400?
    IF JJ=DEROGATE[JJ] THEN
                                                                                00032500?
      FOR I = 1 STEP 1 UNTIL N2 DO
                                                                                00032600?
        W(I, I] = W(I, I) + PERTURB + JJ;
                                                                                000327007
    INVITERATION(N2, W, VECTOR, ITCOUNT, INTCH);
                                                                                00032800?
    FUR I: 1 STEP 1 UNTIL NO DO
                                                                                00032900?
    FUR J:=1.2 DO
                                                                                00033000?
      SHTX[[,JJ]:=VECTOR[[,1]:
                                                                                000331007
  END JJI
                                                                                00033200?
  COMMENT
          SEARCH N COLUMNS OF SMTX CORRESPONDING TO N LIGENVALUES
                                                                                00033300?
           WITH POSITIVE REAL PARTS, SMTX1 AND SMTX2 ARE ITS UPPER
                                                                                00033400?
           AND LOWER HALVES, RESPECTIVELY;
                                                                                00033500?
  INDEX: "OF
                                                                                00033600?
  FOR II=1 STEP 1 UNTIL N2 DO
                                                                                00033700?
    IF EIGR[I] GTR O THEN
                                                                                00033800?
    BEGIN
                                                                                00033900?
      INDEX: #INDEX+1:
                                                                                00034000?
      SEARCH[INDEX]:=I;
                                                                                00034100?
    END:
                                                                                00034200?
```

00034300? 00034400? 00034500? 00034600?

	FO	R	J	1		1	S	T	Ε	P		l	U	N	Ť	1	L	P	١	0	0							
	FO	R	I	1	8	1	S	T	E	P		1	U	N	Ť	İ	L	ŀ	4	D	0							
	8E	G I	N																									
		SM	T	X	1		Ι,	J]	8	m (SM	17	X	1	I		SE	A	R	CH	10	J	1];			
		SM	T	X	2	t I		J	1	8	8 5	SM	ιT	X	ť	N'	•	I	S	E	AF	₹C	Н	[J	3 /	1	
	FN					•	Ĭ								Ĭ													
	ĪN	٧Ł	R	S	Ε	et	٧.	S	М	T	X :	۱,	Ť	E	М	P)	3										
	MU																		()	1								
۸.	D														_					•								

APPENDIX II

ILLIAC IV PROGRAMS

```
THO ILLIAC IV GLYPNIR PROGRAMS, "CLINSYS" AND "INVITERATION" ARE LISTED BELOW. "CLINSYS" IS USED TO OBTAIN THE INVERSE OF A MATRIX
                                                                                00000100?
                                                                                 00000200?
* AND THE SOLUTION OF A COMPLEX SYSTEM OF UNSYMMETRIC LINEAR EQUATIONS,
                                                                                00000300?
8 "INVITERATION" IS USED TO CALCULATE THE PRINCIPAL VECTORS OF DEGREE
                                                                                 00000400?
x K FROM ((W=LAMBDA+I)++K)+XmO.
                                                                                 00000500?
                                                                                 000006002
                                                                                 00000700?
SUBROUTINE CLINSYS, CINT N. CINT R. PCPOINT A. PCPOINT X. PCPOINT B):
                                                                                 000008002
BEGIN & THE SUBROUTINE SOLVES A COMPLEX SYSTEM OF UNSYMMETHIC LINEAR
                                                                                 000009002
      % EQUATIONS AXER. A IS N+2N AND B IS N+2R MATRICES, RESPECTIVELY.
                                                                                00001000?
      * THEIR GENERAL ELEMENTS ARE ACT 2J-13+1+ACT 2J AND BCT 2J-13+
                                                                                 00001100?
      * I_B(I,2J], RESPECTIVELY, R IS THE NUMBER OF RIGHT-HAND SIDES.
                                                                                 000012002
                                                                                 000013002
  SUBROUTINE INNPROD(CREAL C1.CREAL C2.PREAL AK.PREAL BK.CREAL D1.
                                                                                 00001400?
  BEGIN & ACCUMULATES THE SUM OF PRODUCTS AND ADDS IT TO THE INITIAL
                                                                                 00001500?
                                                                                00001600?
         % VALUE (C1, C2).
                                                                                 00001700?
                                                                                 000018002
    CHEAL S12;
    S12; *C1+C2;
                                                                                 000019002
    $121 = $12 + ROWSUM (AK+BK);
                                                                                 00002000?
    D11=5121
                                                                                 000021007
    D21 - S12 - D1;
                                                                                 00002200?
  END!
                                                                                 000023002
                                                                                 00002400?
  SUBMOUTINE CDECOMPOSE(CINT N. PCPOINT A, CREAL DETR, CREAL DETI,
                                                                                 00002500?
                                                                                 00002600?
                           CINT DETE CHPOINT INT);
  BEGIN & THE CUMPLEY UNSYMMETRIC MATRIX A IS DECOMPOSED INTO LU.
                                                                                 000027002
           WHERE L IS LOWER TRIANGULAR AND U IS UNIT UPPER TRIANGULAR
                                                                                 000028000
           MATRICES, AND OVERWRITTEN ON A. A IS STORED IN Nº2N ARRAY
                                                                                 000029002
         3
           AND ITS GENERAL ELEMENTS ARE A[1,2]-11+1+A[1,2]1. INT[1] KEEPS THE RECORD OF ANY INTERCHANGES MADE TO THE ROWS OF
                                                                                 00003000?
                                                                                 000031002
         & A. THE DETERMINANT (DETR+I+DETI)+2++DETE IS ALSO COMPUTED.
                                                                                 000037007
                                                                                 000033002
    CINT I . J . K . L . P . PP;
    PREAL VECTOR AT[64];
                                                                                 000034002
    PREAL ZZI
                                                                                 00003500?
    CHEAL X. Y. Z. V. W. H. HHI
                                                                                 00003600?
                                                                                 000037002
    LABEL FAIL:
                                                                                 00003800?
    MUDE 1 = TRUE;
                                                                                 00003900?
    MUDEI TRUE AND PEN LSS N+NI
                                                                                 00004000?
    LUUP II=1.1.N DO
                                                                                 00004100?
      INNPROD(O,O,A[],A[],INT[],W);
    DETRIBLE DETITEO; DETERED;
                                                                                 00004200?
                                                                                 00004300?
    LUOP KIEL, IN DO
                                                                                 00004400?
    BEGIN
      L1=K1 P1=K+K; PP1=P=11 Z1=0;
                                                                                00004500?
                                                                                 00004600?
      LOOP II=K+1+N DO
                                                                                 00004700?
      BEGIN
                                                                                 00004800?
        MODEI = TRUEI
                                                                                 00004900?
        MODE := TRUE AND PEN LSS N+N;
                                                                                 00005000?
         INNPROD(=GRABONE(A[I],PP=1),O,A[I],AT[PP],H,HH);
                                                                                 00005100?
         INNPROD(=H,=HH,RTL(1,,A[I]),AT(P],X,HH);
                                                                                 00005200?
         INNPROD ("GRARONE (A[I], P"1), O, RTL (1, A[I]), AT(PP), H, HH);
                                                                                 000053002
         INNPROD (H. HH, A[I], AT[P], H, HH);
                                                                                 00005400?
         Y := - H :
                                                                                 00005500?
         MODE I . TRUE !
                                                                                 00005600?
         MODEL TRUE AND PEN EQL PP=1;
                                           A[I]I=XI
                                                                                 00005700?
         MODE = TRUE :
```

```
MODE: TRUE AND PEN EQL P-1;
                                           ACIJ, TY
                                                                                    000058002
          MODE ! TRUE !
                                                                                    000059002
          MODES TRUE AND PENELS
                                                                                    000060007
          XI=(X+X+Y+Y)/INT[]];
                                                                                    00006100?
          IF X GTR Z THEN
                                                                                    000062002
          BEGIN ZIEX; LIEIS ENDS
                                                                                    000063007
       END:
                                                                                    000064002
       MODE : TRUE; HODE : TRUE AND PEN LSS N+N;
                                                                                   000065002
       IF L NEO K THEN
                                                                                    000066002
       BEGIN DETRI--DETRI
                                                                                    000067002
              DETI:==DFTI;
                                                                                    000068002
              ZZ:=A[K];
                                                                                   000069002
              AEK] = AEL JJ
                                                                                   000070002
              A[L] =ZZI
                                                                                   00007100?
              INT[L] := INT[K];
                                                                                   000072002
       END:
                                                                                   000073007
       INT[K] := [ ]
                                                                                   000074002
       XI=GRABONE(A[K],PP=1);
                                                                                   000075007
       Y: =GRABONE(A[K], P=1);
                                                                                   000076007
       21=X+X+Y+Y1
                                                                                   000077007
       WI = X + DETR = Y + DETIS
                                                                                   00007800?
       DET + 1 = X + DETI + Y + DETRI
                                                                                   000079002
       DETRIEW;
                                                                                   0000800002
       IF ABS(DETR) GTR ABS(DETI)
                                                                                   000081002
         THEN WIRDETR ELSE WIRDETIS
                                                                                   000082007
       IF W =0 THEN BEGIN DETE: =0; GO TO FAIL; END;
                                                                                   00008300?
       BEGIN LAREL LI;
                                                                                   000084002
       IF ABS(W) GEQ 1 THEN
L11
       BEGIN W: # W+ 0.06251
                                                                                   00008500?
                                                                                   000086002
             DETI := DETI + 0 . 0625;
                                                                                   00008700?
             DETE ! = DETE + 41
                                                                                   00008800?
             GO TO L1:
                                                                                   000089002
      END:
                                                                                   00009000?
      END:
      BEGIN LABEL L21
                                                                                   000091002
                                                                                   000092007
L21
      IF ABS(W) LSS 0.0625 THEN
                                                                                   00009300?
      BEGIN WIEW+161
                                                                                   00009400?
            DETRI=DETR#16:
                                                                                   00009500?
             DETII=DETI*16;
                                                                                   000094007
             DETER = DETE 41
                                                                                   00009700?
             GO TO L2:
      END:
                                                                                   00009800?
                                                                                   000099002
      ENDI
                                                                                  00010000?
      LOOP J:=K+1.1.N DO
                                                                                  00010100?
      BEGIN
                                                                                  00010200?
        P1=J+J; PP1=P=1;
                                                                                  00010300?
        MODE : TRUE;
        MODE: TRUE AND PEN LSS K+K-2;
                                                                                  00010400?
        INNPROD(=GRABONE(A[K],PP=1),0,A[K],AT[PP],H,HH);
                                                                                  00010500?
        INNPROD(=H,=HH,RTL(1,,A[K]),AT[P],V,HH);
INNPROD(=GRABONE(A[K],P=1),O,RTL(1,,A[PP]),AT[PP],H,HH);
                                                                                  00010600?
                                                                                  00010700?
                                                                                  00010800?
        INNPROD (H, HH, AEK], ATEP], H, HH);
                                                                                  00010900?
        W:=-H:
        MODE : = TRUE :
                                                                                  000110002
        MODE = TRUE AND PEN=PP-13
                                                                                  00011100?
          A[K]; = (V+X+W+Y)/Z;
                                                                                  00011200?
                                                                                  000113007
        MODE = TRUE;
                                                                                  00011400?
```

```
MODE STRUE AND PENSPOLI
                                                                            00011500?
        A(K):=(W+X-V+Y)/Z;
                                                                            00011600?
    END:
                                                                            00011700?
  END;
                                                                            00011800?
FAILI
                                                                            00011900?
END; SOF CDECOMPOSE.
                                                                            00012000?
                                                                            00012100?
SUBHOUTINE CACCSOLVE(CINT N.CINT R.PCPOINT A, PCPOINT AA, CNPOINT P.
                                                                            00012200?
                     POPOINT BAPOPOINT XAPOPOINT BBACINT L);
                                                                            00012300?
BEGIN % SOLVES AX#B, WHERE A IS AN N+2N COMPLEX UNSYMMETRIC AND
                                                                            00012400?
      * B IS AN N+2R COMPLEX MATRICES, RESPECTIVELY. AA IS LU
                                                                            000125002
      * DECOMPOSITION PRODUCED BY "CDECOMPOSE". THE RESIDUALS
                                                                            00012600?
        BB=B-AX ARE ALSO CALCULATED AND AD-BB IS SOLVED, OVER-
                                                                            00012700?
        WRITING O ON BB. L IS THE NUMBER OF ITERATIONS.
                                                                            00012800?
                                                                            00012900?
  SUBROUTINE CSOLVE(CINT N. CINT R. PCPOINT A, CNPOINT INT.
                                                                            000130002
                     PCPOINT B);
                                                                            000131002
                                                                            00013200?
  BLGIN
    CINT I J.K.KK.P.PP;
PREAL VECTOR BT[64];
                                                                            00013300?
    CREAL X.Y.Z.Z1.Z2.H.HH;
                                                                            000135002
    PREAL VECTOR C(64);
                                                                            000136002
                                                                            00013700?
    BOOLEAN AMODE!
    LOUP ITEL TON DO
                                                                            00013800?
      IF INTELL NEG I THEN
                                                                            00013900?
        LOOP J:=R+R.=1.1 DO
                                                                            00014000?
                                                                            00014100?
        BFGIN
          X:=GRABONE(B[I],J=1);
                                                                            00014200?
          MODE = TRUE AND PENBJ#1;
                                                                            00014300?
          B[I]:=GRABONE(B[INT[I]],J=1);
                                                                            00014400?
          B[INT[]];=X;
                                                                            00014500?
                                                                            00014600?
        END:
    LOOP K = R+R = 2,2 DO
                                                                            00014700?
                                                                            00014800?
    BEGIN
                                                                            00014900?
      KK1=K=11
                                                                            00015000?
      LOOP I:=1.1.N 00
                                                                            00015100?
      BEGIN
                                                                            00015200?
        MODE:=TRUE AND PEN LSS I+1-2;
                                                                            00015300?
        INNPROD(=GRABONE(B[I]*KK=1),O*A[I]*BT[KK]*H*HH);
                                                                            00015400?
        INNPROD(-H,-HH,RTL(1,A[[]),BT(K],X,HH))
                                                                            00015500?
        INNPROD(=GRABONE(B[I]+K=1)+O+RTL(1++A[I])+BT(KK)+H+HH);
                                                                            00015600?
        INNPROD(H, HH, A[I], BT(K], H, HH);
                                                                            00015700?
        Y : = - H :
        P:=I+I; PP:=P=1;
                                                                            00015800?
                                                                            00015900?
        Z11=GRABONE(A[]],PP=1);
                                                                            00016000?
        Z2 = GRABONE (A[I] ,P=1);
                                                                            00016100?
        L1=Z1+Z1+Z2+L21
                                                                            00016200?
        MODE:=TRUE AND PEN=KKJ
                                                                            00016300?
          B[]]:=(X+Z1+Y+Z2)/4;
                                                                            00016400?
        MODE := TRUE AND PEN=K;
                                                                            00016500?
          B[[]:=(Y+Z1=X+Z2)/Z;
                                                                            00016600?
      END;
                                                                            00016700?
      LOOP I:=N,=1,1 00
                                                                            00016800?
      BEGIN
                                                                            00016900?
        AMODE: = TRUE AND PEN GEQ N;
                                                                            00017000?
        MODE : = AMODE :
                                                                            00017100?
        INNPROD (-GRABONE (B[]] , KK-1), O, A[]], BT[KK], H, HH);
```

```
INNPROD(-H,-HH,RTL(1,,A[]]),BT[K],H,HH);
                                                                                 00017200?
                                                                                 00017300?
           MODE: TRUE AND PENSKK-11 C[1] + 4 H J
                                                                                 00017400?
           MODE: MANODE:
           INNPROD(-GRABONE(B[I],K-1),O,RTL(1,,A[I]),BT[KK],H,HH);
                                                                                 000175002
           INNPROD(H, HH, A[I], BT[K], H, HH);
                                                                                 000176002
          MODE: TRUE AND PENEK-1; C[]:=++;
MODE: TRUE AND (PENEK-1 OR PENEK-1); B[]:=C[];
                                                                                 000177002
                                                                                 000178002
                                                                                 000179002
        END:
                                                                                 000180002
      END:
                                                                                 000181002
    END: & OF CSOLVE.
                                                                                 00018200?
    CINT I.J.K.D2.C.CC;
                                                                                 000183007
    CHEAL E, DO, D1 , XHAX , BBNAX , EPS , E1 , E2 , H , HH;
                                                                                 00018400?
                                                                                 000185002
    PREAL VECTOR XT[64]:
                                                                                 00018600?
    AUDLEAN AMODES
                                                                                 00018700?
    LABEL LZOTLLS
    EPS:=1:00-10;
                                                                                 000188007
                                                                                 000189002
    AMODE, #PEN GEQ No
                                                                                 00019000?
    MUDE : # AMODE ;
                                                                                 00019100?
    LUOP I 1 #1 + 1 + N OO
                                                                                 00019200?
    BEGIN XEIJ := 0.0;
           BB[[]:*B[[]:
                                                                                 000193002
    END:
                                                                                 000194007
    L1=01 D01=01
                                                                                 00019500?
    MUDE != TRUE !
                                                                                 000196007
                                                                                 000197002
L3: CSULVE(N.R.AA.P.BB);
    LI=L+1; D2;=0; D1;=0;
                                                                                 000198007
                                                                                 000199002
    MODE ##AMODE !
    LUOP I:=1:1:N DO
                                                                                 00020000?
      I[]]88+[]]X#I[]]X
                                                                                 00020100?
    LUOP JEE1:1:R DO
                                                                                 00020200?
    BEGIN
                                                                                 000203007
      XMAXImOI
                BBMAX =01
                                                                                 00020400?
      C:=J+J; CC:=C=1;
                                                                                 000205007
      LOOP I = 1 . 1 . N DO
                                                                                 00020600?
      BEGIN
                                                                                 000207007
        E11=GRABONE(X[I],CC=1); E21=GRABONE(X[I],C=1);
                                                                                 00020800?
        E:=E1*E1+E2*E21
                                                                                 00020900?
        IF E GTR XMAX THEN XMAXIBE;
                                                                                 000210007
        E11=GRABONE(BB[]).CC); E21=GRABONE(BB[]).C);
                                                                                 00021100?
        E:=E1+E1+E2+E2;
                                                                                 000212007
        IF E GTR BBMAX THEN BBMAX : #E }
                                                                                 000213007
        MODE = TRUE :
                                                                                 00021400?
        INNPROD (=GRABONE(B[I],CC),O,A[I],XT[CC],H,HH);
                                                                                 00021500?
        INNPRODUCHO "HO RTL(100A(I)) XT[C] + H+H);
                                                                                 000216007
        MODE: TRUE AND PENECC-13 BB[1] ++ H;
                                                                                 00021700?
        MODELETRUEL
                                                                                 00021800?
         INNPROD( GRABONE (B[I], C), O, RTL(1, A[I]), XT[C], H, HH);
                                                                                 00021900?
         INNPROD(H*HH*A[I]*XT[c]*H*HH);
                                                                                 00022000?
         MODE ** TRUE AND PEN*C=1; BB[] : = +H;
                                                                                 00022100?
      END:
                                                                                 000222007
      IF BBMAX/XMAX GTR D1 THEN D1; #BBMAX/XMAX;
                                                                                 00022300?
      IF BBMAX GTR (2+EPS)+(2+EPS)+XMAX THEN D21+11
                                                                                 00022400?
    ENDI
                                                                                 00022500?
    IF D1 GTR D0+0.25 AND L NEQ 1 THEN GO TO ILL;
                                                                                 00022600?
    DU:=D11
                                                                                 000227007
    IF D2=1 THEN GO TO L3;
                                                                                 00022800?
```

```
ILLI
                                                                              000229002
  END! & OF CACCSOLVE.
                                                                              00023000?
                                                                              00023100?
                                                                              00023200?
  PREAL VECTOR AA, BB16411
  CREAL VECTOR INTCHE6411
                                                                              00023300?
  CINT INNORDETERITE
                                                                              00023400?
  CREAL DETRODETIS
                                                                              00023500?
  LOOP II . 1 . N DO
                                                                              00023600?
    AAEIJ: #AEIJ3
                                                                              000237002
                                                                              00023800?
  CDECOMPOSE(N, AA, DETR, DETI, DETE, INTCH);
  CACCSOLVE(N.R.A.AA, INTCH.B.X.BB.IT);
                                                                              00023900?
ENDI & OF CLINSYS.
                                                                              00024000?
                                                                              000241002
                                                                              00024200?
SUBROUTING INVITERATION (CINT No PCPOINT WO PCPOINT XO CINT KK):
                                                                              00024300?
REGIN & THE SUBROUTINE COMPUTES THE EIGENVECTORS AND PHINCIPAL
                                                                              000244002
                                                                              00024500?
      * VECTORS BY SOLVING ((W=LAMBDA+I)++K)+X=0. THE SUBRUUTINES
        "CDECOMPOSE" AND "CACCSOLVE" IN "CLINSYS" ARE USED.
                                                                              000246007
  PE HEAL SUBROUTINE SORT AS RGA (PE REAL ARG AS RGA) !
                                                                              00024700?
                                                                              00024800?
  BEGIN
                                                                              00024900?
   CALL SQRT64()1
  END; & OF SORT.
                                                                              000250002
                                                                              00025100?
    CINT I JIII DETE RILI
                                                                              00025200?
    CHEAL DETR.DETI.MAXVAL, XX1, XX2;
                                                                              00025300?
    PREAL VECTOR W2.8. 82.88. XX[64],
                                                                              00025400?
    CHEAL VECTOR P.ROOT[64]
    PHEAL TRISTRES
                                                                              00025500?
                                                                              00025600?
    LABEL INVITONEXTOFINIST
                                                                              000257002
      MODE := PEN LSS N+N;
      LOOP ITEL TON DO WELLIEW(I);
                                                                              00025800?
      CDECOMPOSE(N, W2, DETR, DETI, DETE, P);
                                                                              00025900?
                                                                              000260007
      MODE := PEN LSS 21
                                                                              000261002
      LOUP I := 1 . 1 . N DO
      BEGIN B2[I]:=1,0; B[I]:=B2[I]; END;
                                                                              00026200?
                                                                              00026300?
      KK 1 = 1 1
                                                                              00026400?
      R1=11
                                                                              00026500?
INVITE
                                                                              000266007
      MODEL PEN LSS N+N1
                                                                              00026700?
      CACCSOLVE(N.R.W.W2.P.B.X.BB.L);
                                                                              00026800?
      MODFIEPEN LSS 21
                                                                              000269002
      LOOP 1:=1:1:N DO
                                                                              00027000?
      BEGIN
                                                                              00027100?
        ROOT[I] = ROWSUM(X[I] + X[I]);
                                                                              000272007
        ROOT[I] = SQRT(ROOT[I]);
                                                                              00027300?
      ENDI
                                                                              00027400?
      MAXVALI=ROOT[11]
                                                                              00027500?
      LOOP I 1=2 1 N DO
           MAXVAL GEO HOOT[1] THEN MAXVAL = MAXVAL ELSE MAXVAL = ROOT[1] 00027600?
                                                                              00027700?
      LOOP ItalalaN DO
        IF ROOT[ ] = MAXVAL THEN BEGIN II : # 1 GO TO NEXT ENU ;
                                                                              00027800?
                                                                              00027900?
NEXTI
                                                                              00028000?
      XX1, = GRABONE(X[II] = 0);
                                                                              00028100?
      XX2:==GRABONE(X[II],1);
                                                                              00028200?
      IF -XX2 NEQ O THEN
                                                                              00028300?
      BEGIN
                                                                              00028400?
        LOOP 1:=1 -1 -N DO
                                                                              00028500?
        BEGIN
```

```
MODE: PEN LSS 2:
TR1: GRABONE(X[]].0) *XX1 GRABONE(X[]].1) *XX2;
                                                                                    00028600
                                                                                    00028700
              TR21#GRABONE(X[I],0)+XX2+GRABONE(X[I],1)+X41;
                                                                                    00028800
           MODE: #PEN#01
                           XX[I] | WTR1:
                                                                                    00028900
                           XX[[]:=TR2;
                                                                                    00029000
           HODE: #PEN=1;
         END:
                                                                                    00029100
         MODE := PEN LSS 21
                                                                                    00029200
         XX1: GRABONE (XX[II].0);
                                                                                    000293001
         LOUP INMINION DO XIIII MXXIII/XXII
                                                                                    00029400
       END
                                                                                    000295001
       ELSE LOOP Isate 1. N DO XIIJ : aXIIJ/XX11
                                                                                    000296001
       LOOP I := 1 . 1 . N DO
                                                                                    00029700
       BEGIN
                                                                                    000298001
         IF ABS(B2[I]-X[I]) GTR 0-10 THEN
                                                                                    000299001
         BEGIN
                                                                                    00030000%
           IF KK GTR 10 THEN GO TO FINIS;
LOOP J:m1,1,N DO BEGIN B[I]:mX[I];
                                                                                    000301001
                                                                                    000302001
                                      B2[[]:=X[[];
                                                                                    000303001
                               ENDI
                                                                                    000304001
           KKI=KK+11
           GO TO INVITE
                                                                                    00030600%
                                                                                    000307001
         END
         ELSE GO TO FINIS!
       END:
                                                                                    000309001
FINISI
                                                                                    000310001
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Security Classification DOCUMENT CONTROL DATA - R & D (Security classification of title, body of abetract and indexing annotation must be entered when the overall report is classified) ORIGINATING ACTIVITY (Corporate author) 20. REPORT SECURITY CLASSIFICATION Center for Advanced Computation UNCLASSIFIED University of Illinois at Urbana-Champaign 26. GROUP Urbana, Illinois 61801 3. REPORT TITLE A NUMERICAL SOLUTION OF THE MATRIX RICCATI EQUATIONS 4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Research Report 5. AUTHOR(5) (First name, middle initial, last name) Killion Noh S. REPORT DATE 78. TOTAL NO. OF PAGES 75. NO. OF REFS January 20, 1972 16 SE. CONTRACT OR GRANT NO. M. ORIGINATOR'S REPORT NUMBER(S) DAHC04 72-C-0001 b. PROJECT NO. CAC Document No. 24 ARPA Order 1899 Sb. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) UIUCDCS-R-72-498 10. DISTRIBUTION STATEMENT Copies may be obtained from the address given in (1) above. Distribution unlimited; approved for public release. 11. SUPPLEMENTARY NOTES 12. SPONSORING MILITARY ACTIVITY U.S. Army Research Office-Durham None Durham, North Carolina

13. ABSTRACT

The eigenvector solution of the time-invariant matrix Riccati equation is discussed. The coefficient matrix of the canonical equation is allowed to have multiple eigenvalues, namely, the matrix could be either derogatory or defective. The solution of matrix Riccati equation is then calculated from a part of similarity transformation which should reduce the coefficient matrix to the Jordan canonical form.

Security Classification	LI	IK A	LIN	K B	LIN	K C
KEY WORDS	ROLE		ROLE	WT	ROLE	WT
Ordinary and Partial Differential Equations						
Matrix Algebra						
Nonlinear and Functional Equations						
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SHEET 010CDCS-R-72-498	3. Recipient's Accession No.									
4. Title and Subtitle A NUMERICAL SOLUTION OF THE MATRIX RICCATI EQUATIONS	5. Report Date January 20, 1972									
	6.									
7. Author(s) Killion Noh 8. Performing Organizati										
9. Performing Organization Name and Address	10. Project/Task/Work Unit No.									
Center for Advanced Computation										
University of Illinois at Urbana-Champaign	11. Contract/Grant No.									
Urbana, Illinois 61801	DAHCO4 72-C-0001									
12. Sponsoring Organization Name and Address	13. Type of Report & Period Covered									
U.S. Army Research Office-Durham										
Duke Station	Research									
Durham, North Carolina	14.									
15. Supplementary Notes										
None										

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17. Key Words and Document Analysis. 170. Descriptors

Ordinary and Partial Differential Equations

Matrix Algebra

Nonlinear and Functional Equations

17b. Identifiers/Open-Ended Terms

17c. COSATI Field/Group

18. Availability Statement Copies may be obtained from the address in (9) above.

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19.	Security Class (This
	Report)
	UNCLASSIFIED
20.	Security Class (This

21. No. of Pages 38 22. Price

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